Frictional Damping

Daniel Riddoch, Alice Cicirello, David Hills

Department of Engineering Science

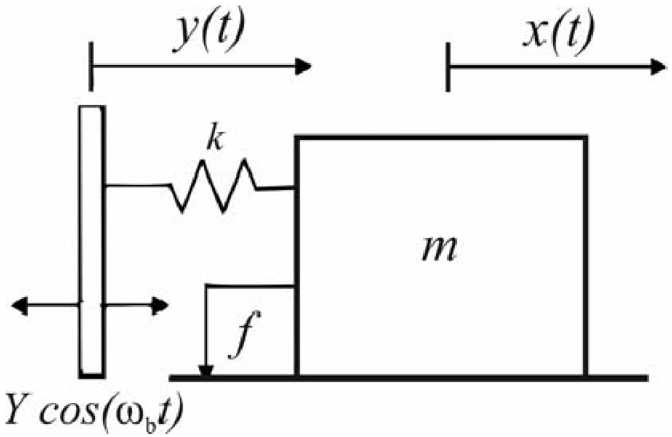
University of Oxford

January 2018

Frictional damping in the *sliding* Regime

- Non-linear effects which can lead to unexpected failures
- Example: effects on vibration of a fan blade (dovetail joint) and other frictional interfaces of engines. These include other blade retention (firtrees/dovetails in compressor/turbine, and general casing damping, e.g. by bolted flange joints.

Rigid Body Motion Problem (sliding)

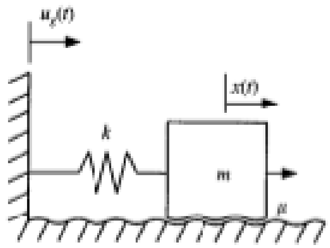


- No analytical model for steady state behaviour in this problem

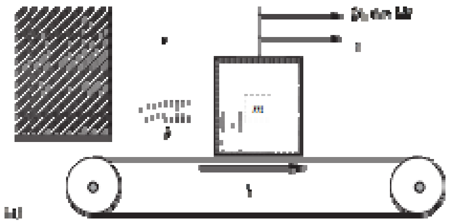
What we know about frictional damping models

- Previous work considers
  - Moving belt problem
  - Combined Coulomb and viscous damping
  - Direct force application to mass
- No consideration of exactly our problem

Direct application of force to mass and base excitation



Direct application of force to mass



Moving belt problem

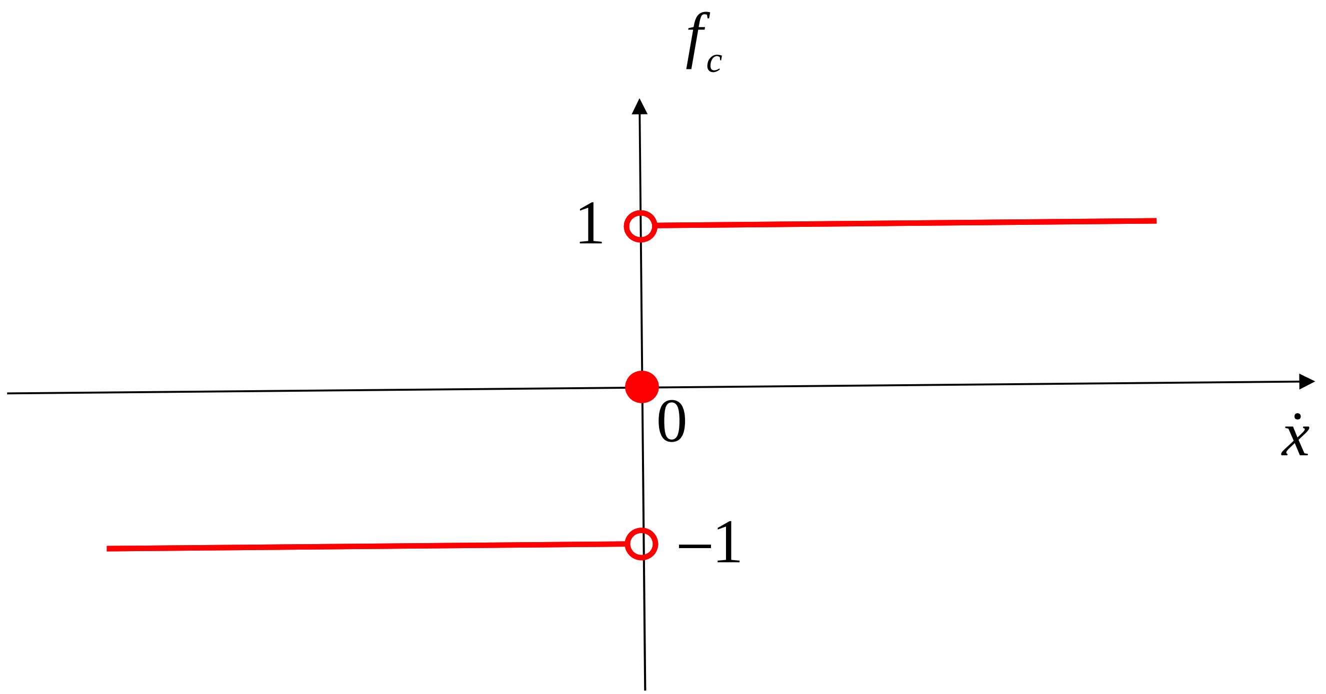
Coulomb friction oscillator: modelling and responses to harmonic loads and base excitations, Hong and Liu, 1999

Frictional contact

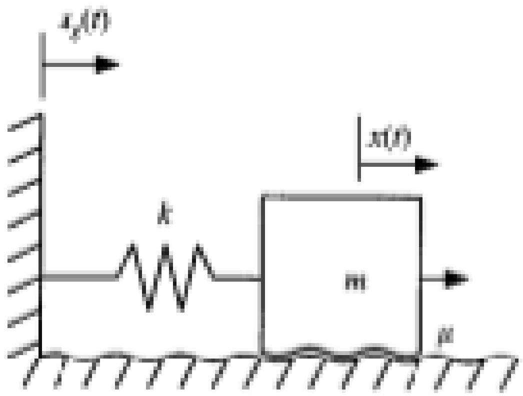
Coulomb model:

- simplest model
- widely used
- Static and dynamic friction is assumed equal

$$f_c = \begin{cases} -1 & \dot{x} > 0 \\ 0 & \dot{x} = 0 \\ 1 & \dot{x} < 0 \end{cases}$$



Formulation



Force in spring ↓ Friction force ↓

- Equation of motion

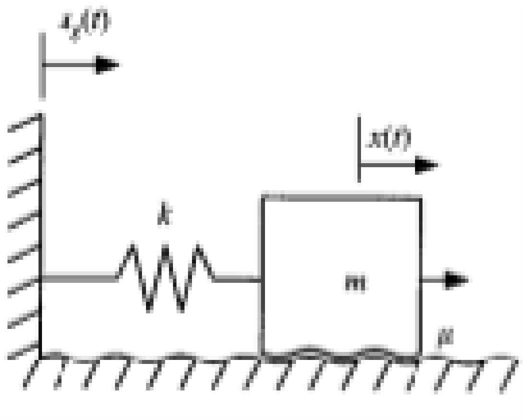
$$m\ddot{x}(t)+k\left[x(t)-Y\cos(\omega_b t)\right]=\mu mg \quad \dot{x}<0$$

$$m\ddot{x}(t)+k\left[x(t)-Y\cos(\omega_b t)\right]=-\mu mg \quad \dot{x}>0$$

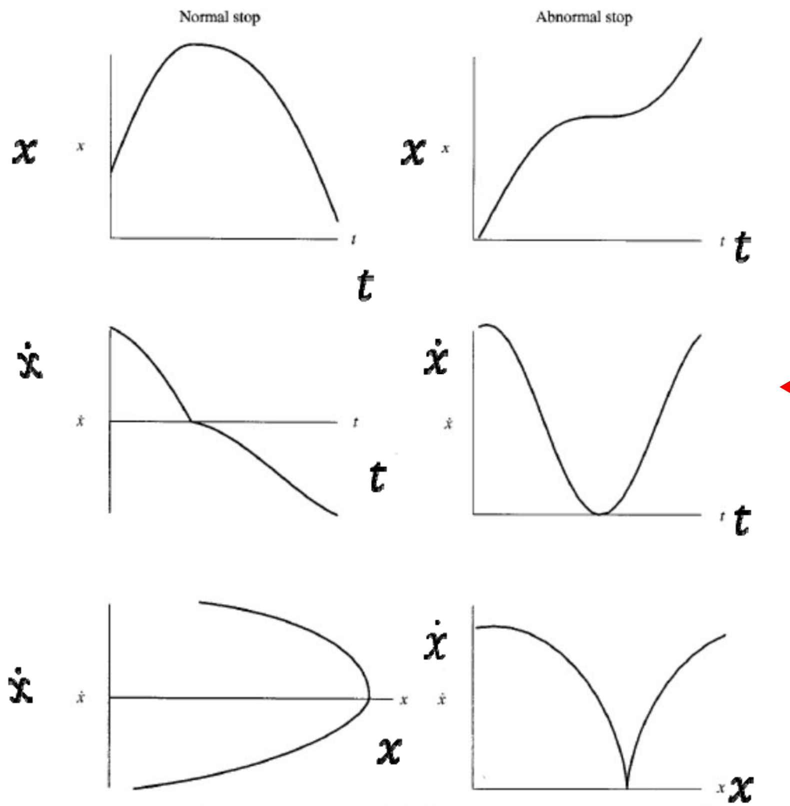
- If $\mathrm{k}|y-x|>\mu mg \Rightarrow acceleration. If\ \mathrm{k}|y-x|<\mu mg \nRightarrow stationary$
- Instantaneous "normal" stop criteria (Hong and Liu, 1999):
  - force on the spring being greater or equal (in magnitude) to the limiting friction
  - Velocity is equal to zero
  - Frictional force times acceleration must be less than zero
- Instantaneous "abnormal" stop criteria (Hong and Liu, 1999):
  - Velocity is equal to zero
  - Frictional force times acceleration must be greater than zero

Coulomb friction oscillator: modelling and responses to harmonic loads and base excitations, Hong and Liu, 1999

What we know



1188 Normal H.-K. HONG AND C.-S. LIU Abnormal

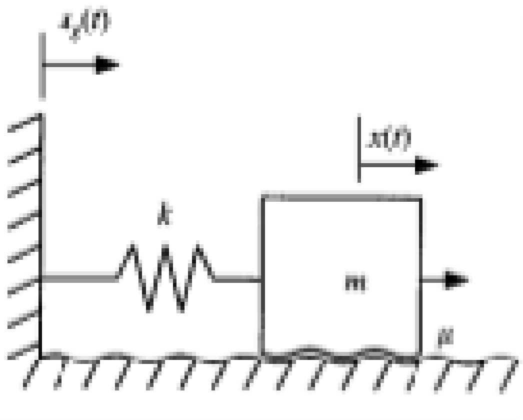


At turning points →

← At intermediate points

Figure 14. Stops with zero duration are divided into two types: normal stops and abnormal stops.

Coulomb friction oscillator: modelling and responses to harmonic loads and base excitations, Hong and Liu, 1999



What we know

- Stops of finite duration conditions:
  - Velocity is zero
  - Spring force (magnitude) less than limiting friction value

- Start-to-slide conditions:

$$t_{slide} = \begin{cases} t_i & \text{if } b_1 > 1 \text{ and } b_2 < -1 & \text{(a)}, \\ \infty & \text{if } b_1 < -1 \text{ or } b_2 > 1 & \text{(b)}, \\ \frac{\arcsin b_1}{\omega_d} & \text{if } 0 < b_1 \leqslant 1 \text{ and } b_2 < -1 & \text{(c)}, \\ \frac{\arccos b_1}{\omega_d} & \text{if } b_2 < -1 \leqslant b_1 \leqslant 0 & \text{(d)}, \\ \frac{\arcsin b_2}{\omega_d} & \text{if } 0 < b_2 \leqslant 1 < b_1 & \text{(e)}, \\ \frac{\arccos b_2}{\omega_d} & \text{if } b_1 > 1 \text{ and } -1 \leqslant b_2 \leqslant 0 & \text{(f)}, \\ \min(t_1, t_2) & \text{if } -1 \leqslant b_2 < b_1 \leqslant 1 & \text{(g)}, \end{cases} \qquad (30\text{–}36)$$

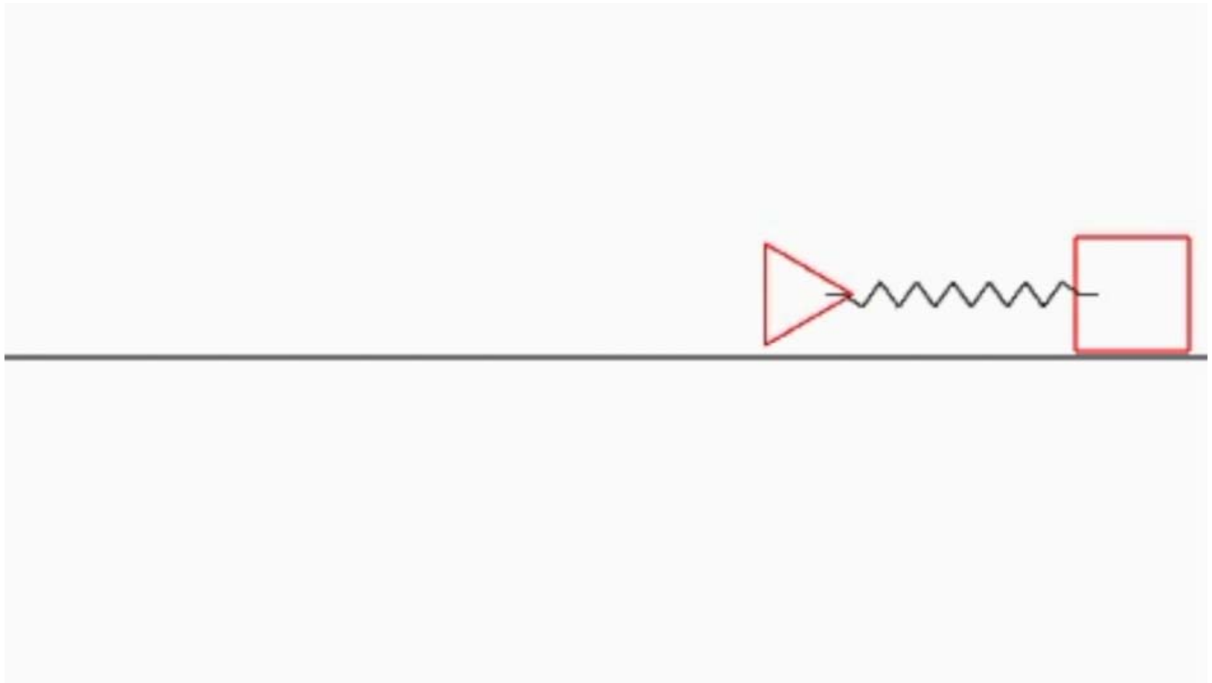
$$b_1 := \frac{kx(t_i) + r_y}{p_0}, \quad b_2 := \frac{kx(t_i) - r_y}{p_0}, \quad b_1 > b_2,$$

where

$$t_j = \begin{cases} \frac{\arcsin b_j}{\omega_d} & \text{if } b_j > 0, \\ \frac{\arccos b_j}{\omega_d} & \text{if } b_j < 0 \end{cases}$$

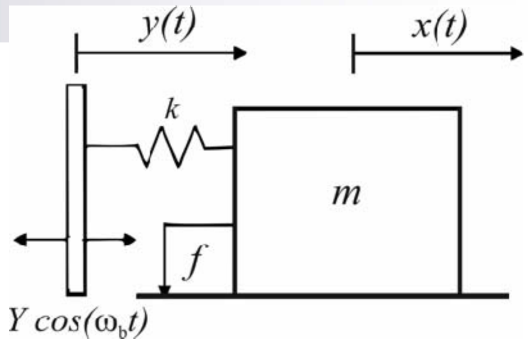
Coulomb friction oscillator: modelling and responses to harmonic loads and base excitations, Hong and Liu, 1999

Example Response:



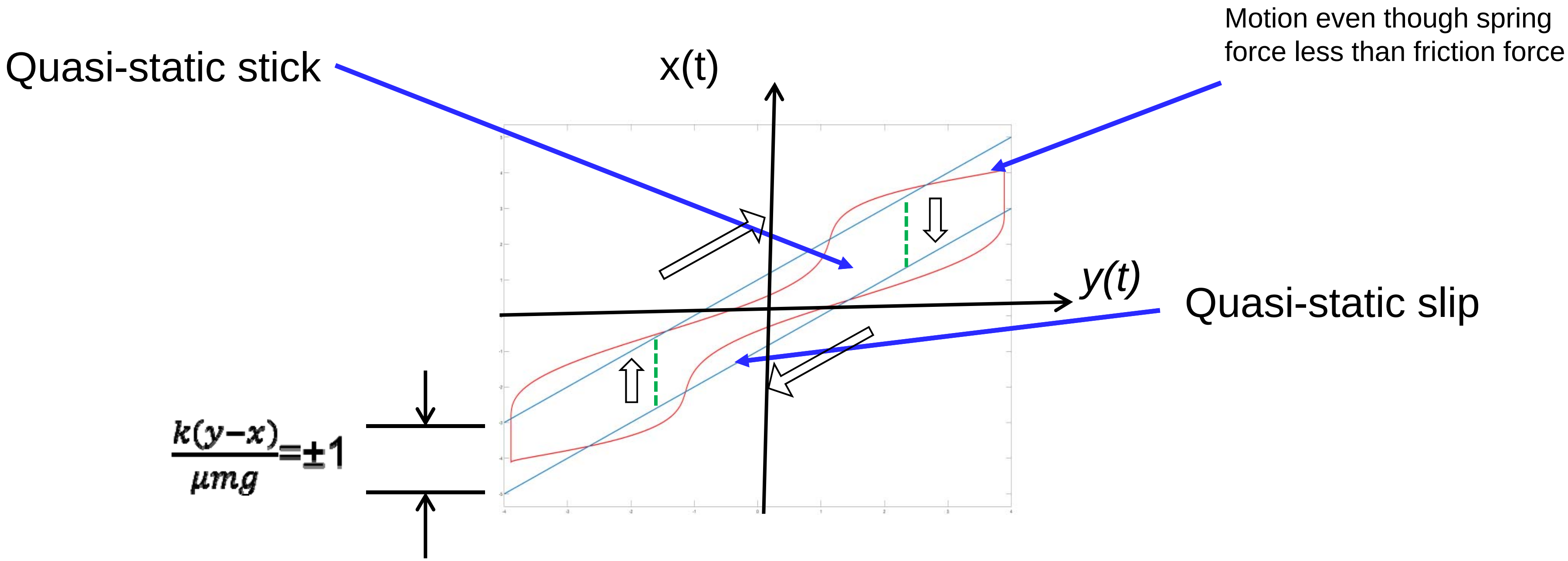
Animation showing the response of a simple case

Quasi-static & Dynamic Responses

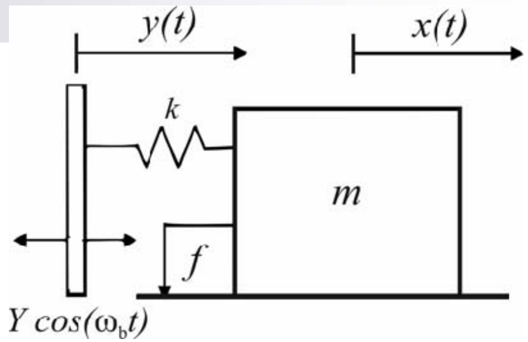


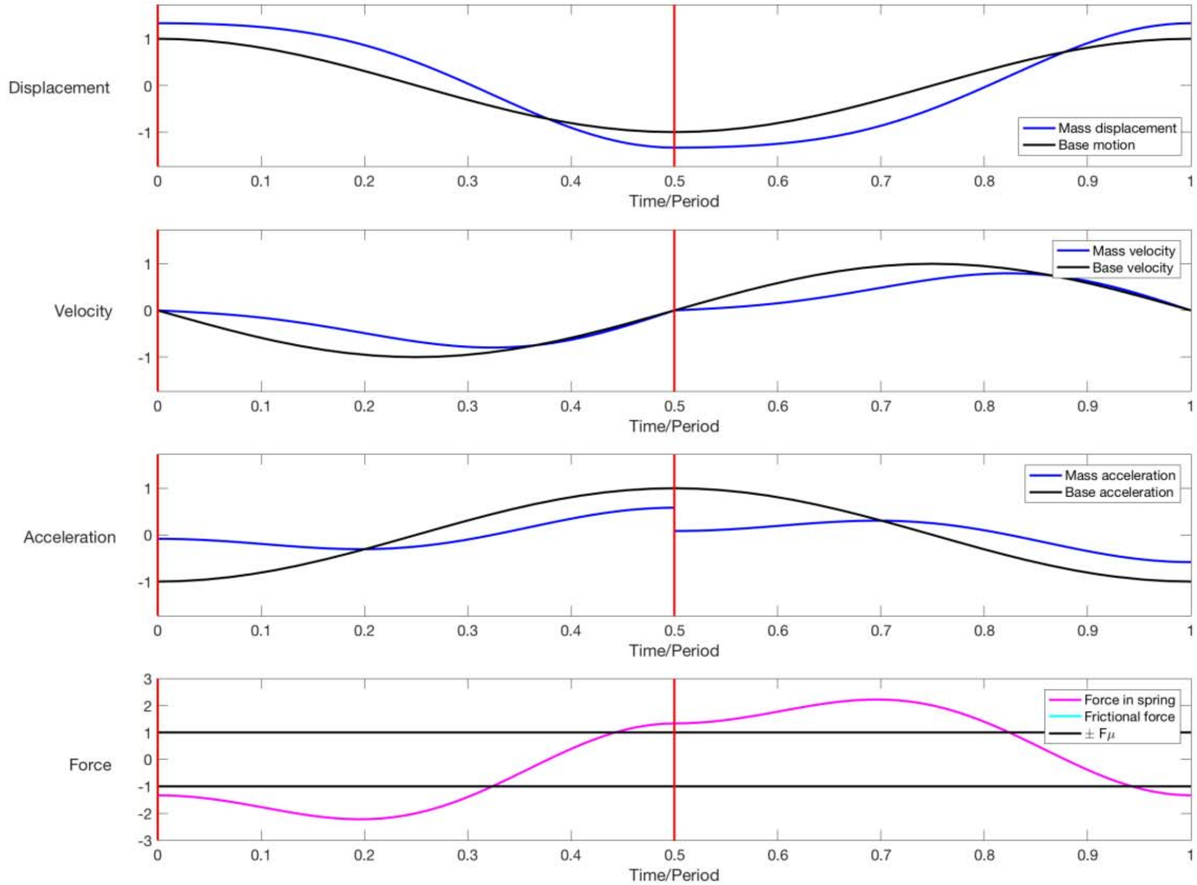
------------ Quasi static Response

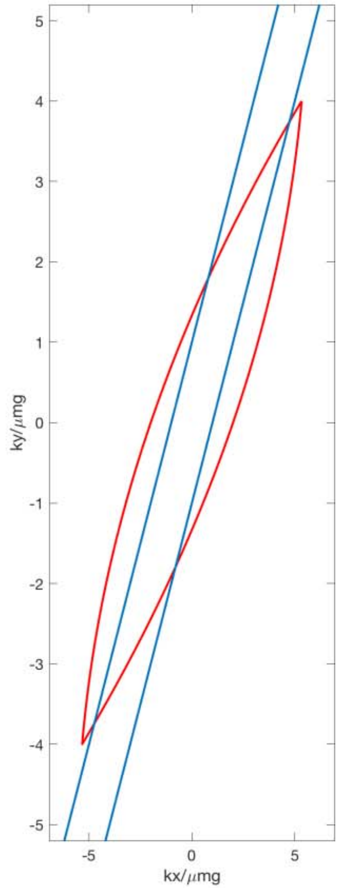
——— Dynamic Response



Low friction – ‘normal’ stops at ends only

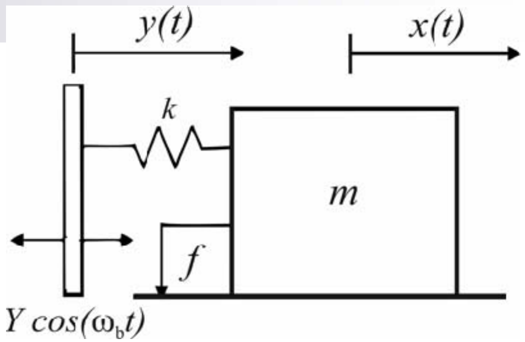


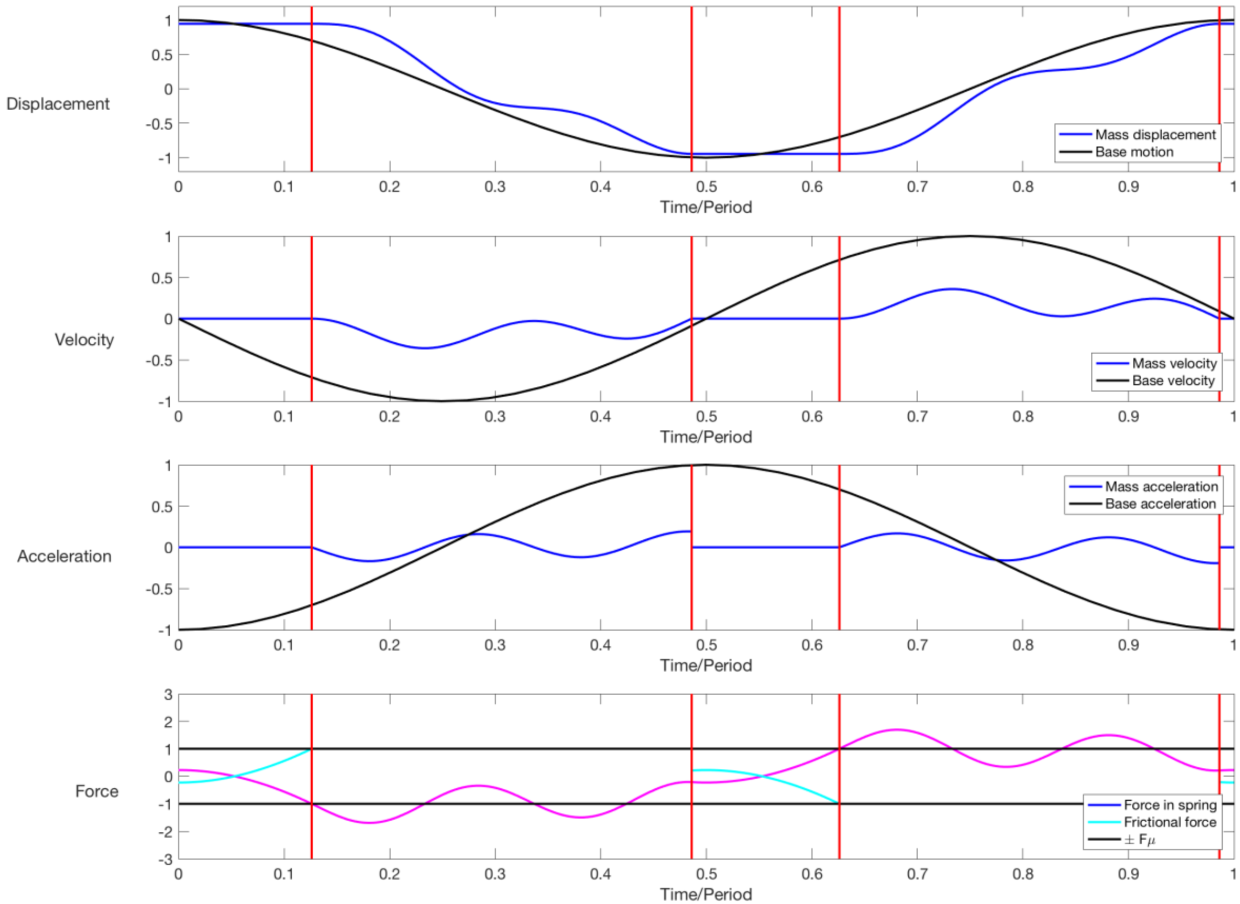


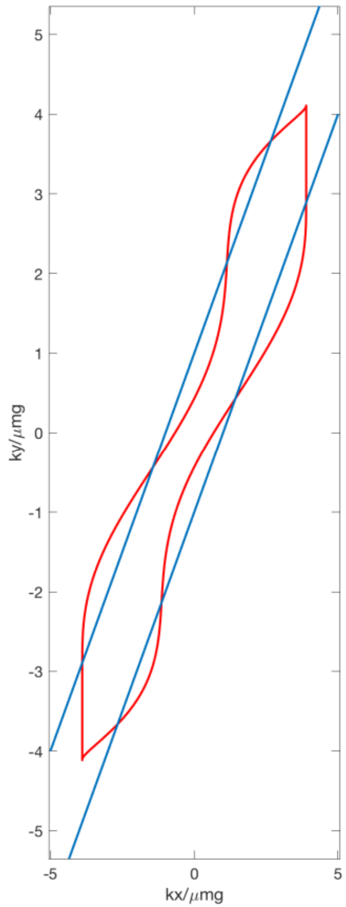


Spring force is greater than friction when velocity reaches zero

Higher friction – ends stops + intermediate stops

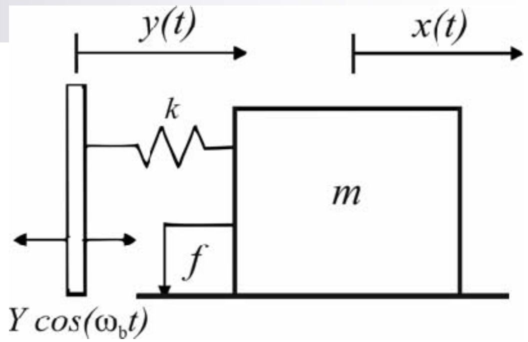






Velocity is non-zero for all time when spring force is less than friction

Solution: for periods of Motion



- Solution for each phase

$$x(t) = A_1\cos(\omega_n t) + B_1\sin(\omega_n t) + \frac{\mu g}{\omega_n^2} + \frac{\omega_n^2 Y}{\omega_n^2 - \omega_b^2}\cos(\omega_b t) \qquad \dot{x} < 0$$

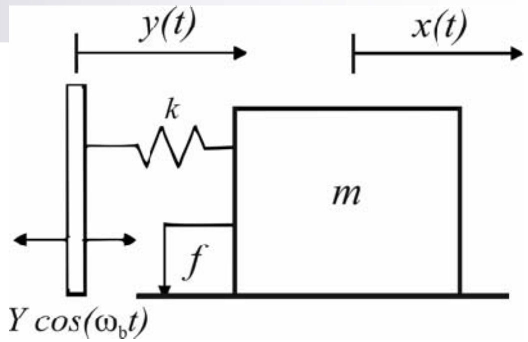
$$x(t) = A_2\cos(\omega_n t) + B_2\sin(\omega_n t) - \frac{\mu g}{\omega_n^2} + \frac{\omega_n^2 Y}{\omega_n^2 - \omega_b^2}\cos(\omega_b t) \qquad \dot{x} > 0$$

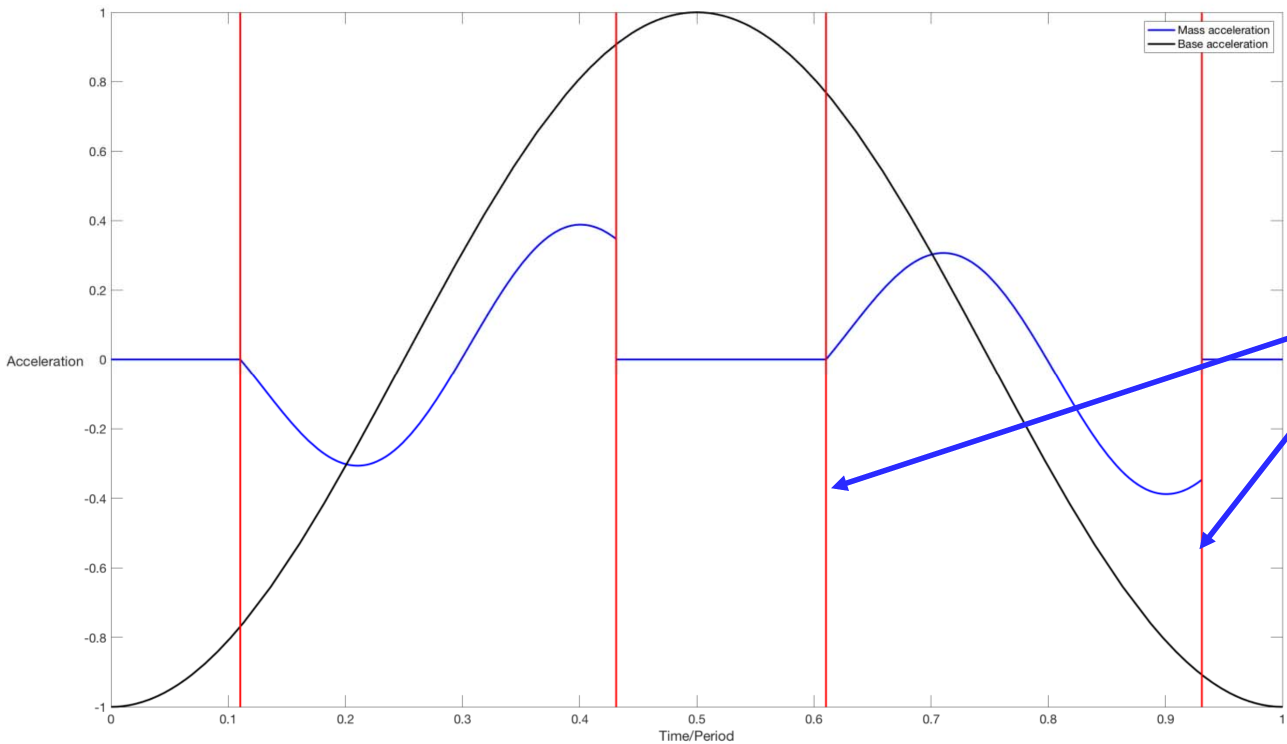
- Using boundary conditions to obtain the solution for a steady state cycle directly

1. $x_1(\tau_0) = x_2(\tau_0 + \delta + \frac{T}{2})$
2. $x_1(\tau_0 + \delta) = x_2(\tau_0 + \frac{T}{2})$
3. $x_1(\tau_0) + x_2(\tau_0 + \frac{T}{2}) = 0$
4. $x_1(\tau_0 + \delta) + x_2(\tau_0 + \delta + \frac{T}{2}) = 0$
5. $\dot{x}_1(\tau_0) = 0$
6. $\dot{x}_1(\tau_0 + \delta) = 0$
7. $\dot{x}_2(\tau_0 + \frac{T}{2}) = 0$
8. $\dot{x}_2(\tau_0 + \delta + \frac{T}{2}) = 0$
9. $\ddot{x}_1(\tau_0) = 0$
10. $\ddot{x}_2(\tau_0 + \frac{T}{2}) = 0$
11. $F_{spring}(\tau_0) = \mu m g$
12. $F_{spring}(\tau_0 + \frac{T}{2}) = \mu m g$

Mass acceleration

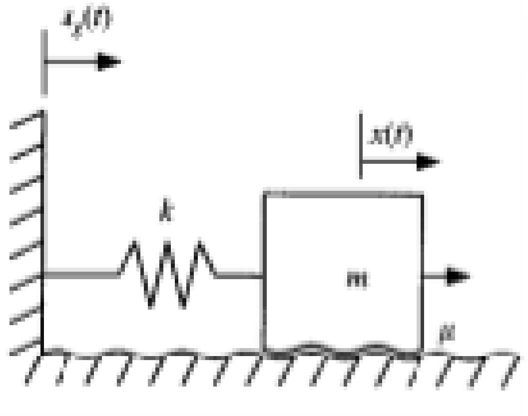
Base acceleration





Discontinuity in acceleration

What was puzzling



- Found asymmetric response to symmetric loading
- Found odd number of stops

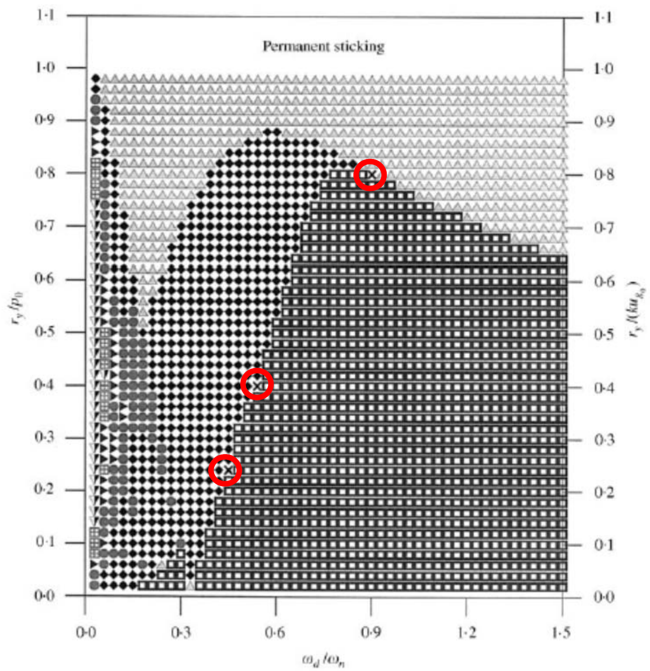
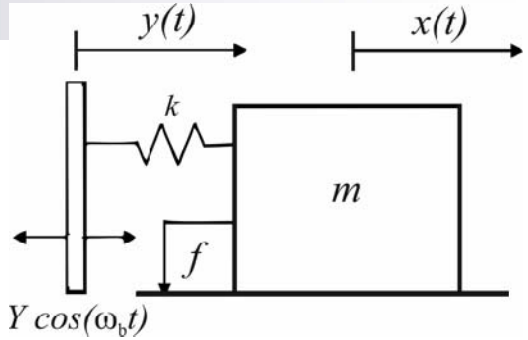


Figure 8. The distribution of the nine types of motions: permanent sticking, zero stop per cycle (i.e., non-sticking oscillation) one stop per cycle and 2, 4, 6, 8, 10 and more stops per cycle in the parametric plane $(1/\alpha, \Omega)$. $1/\alpha = r_y/p_0$, $\Omega = \omega_d/\omega_n$: □, zero stop; ×, one stop; △, two stops; ◆, four stops; ●, six stops; ▶, eight stops; ⊞, ten stops; ∇, more stops.

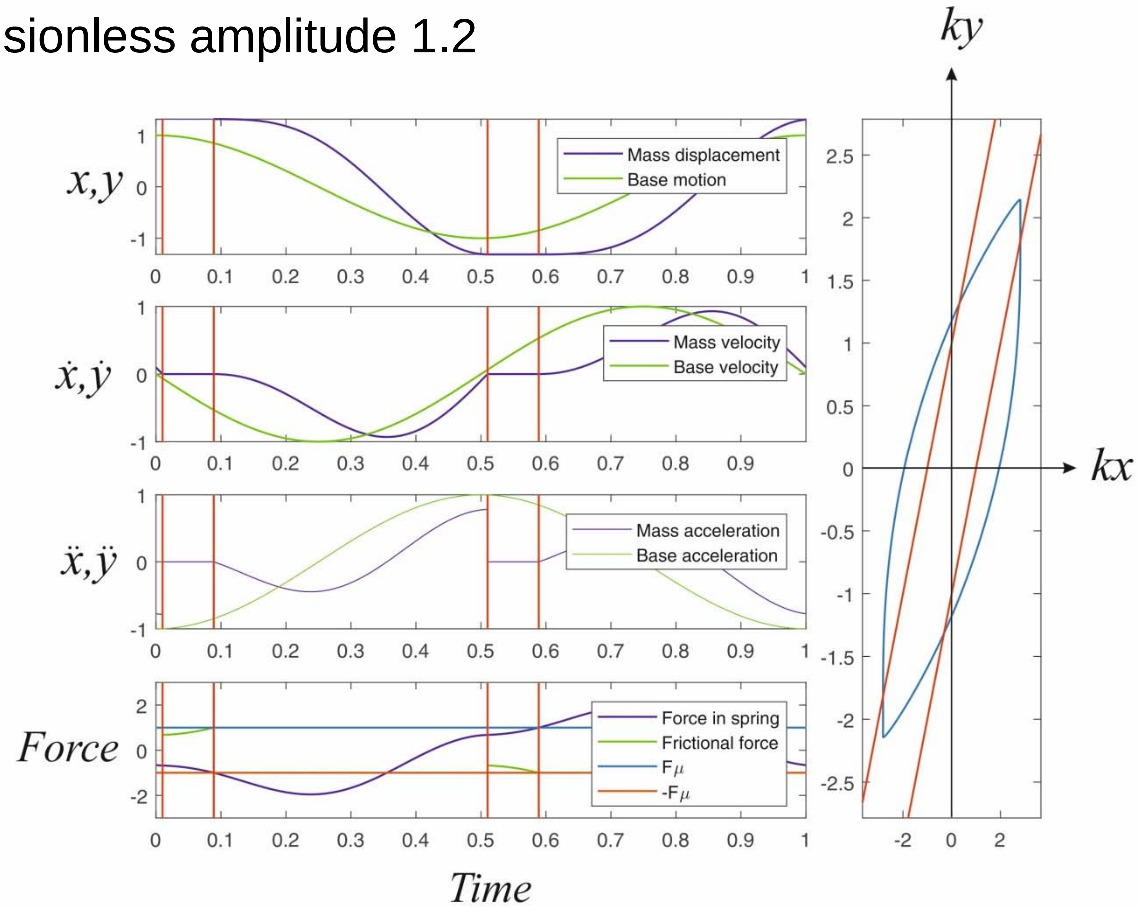
- Focused mainly on low frequency ratios

Coulomb friction oscillator: modelling and responses to harmonic loads and base excitations, Hong and Liu, 1999

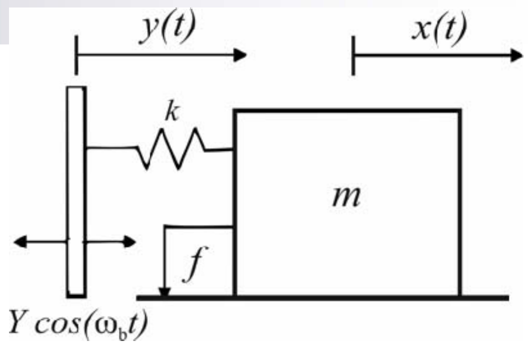
Results



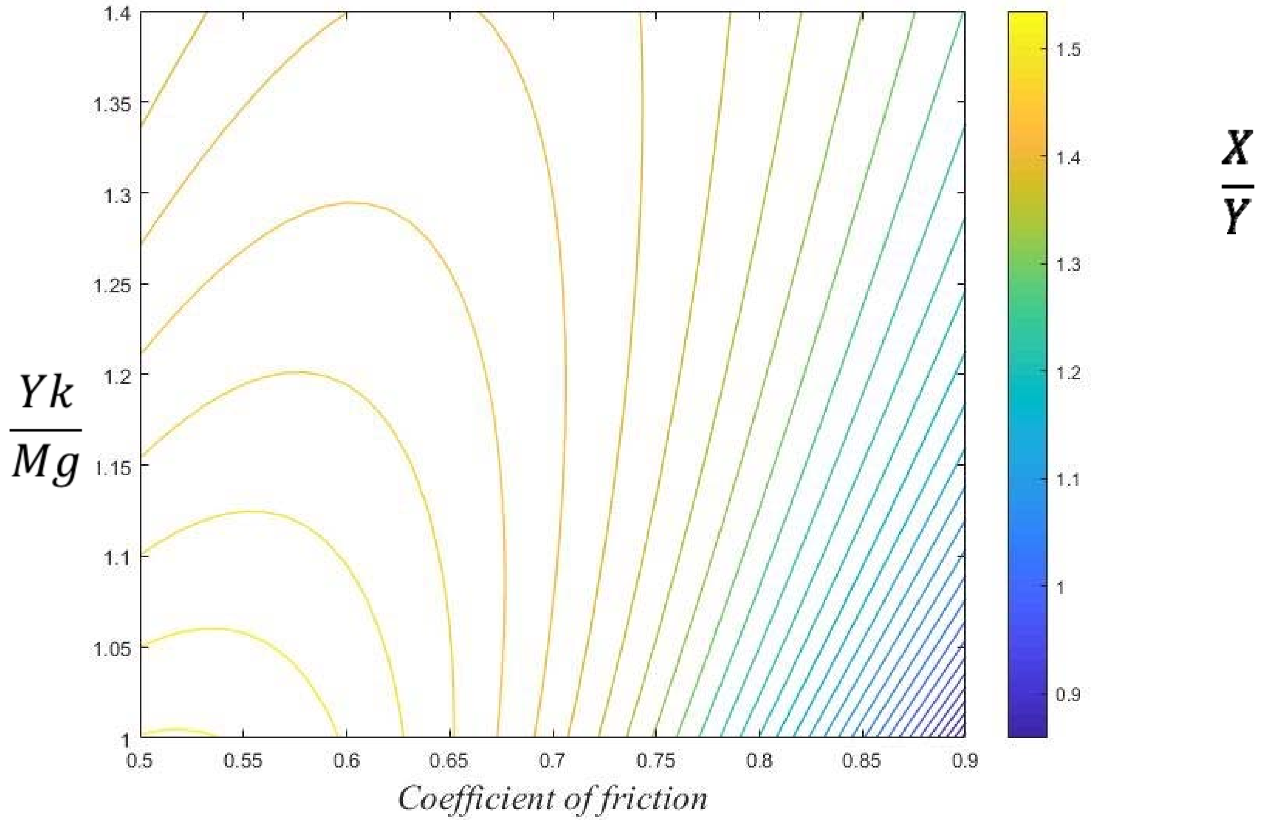
- Results from a single case
  - Frequency ratio 0.5
  - Coefficient of friction 0.5
  - Dimensionless amplitude 1.2



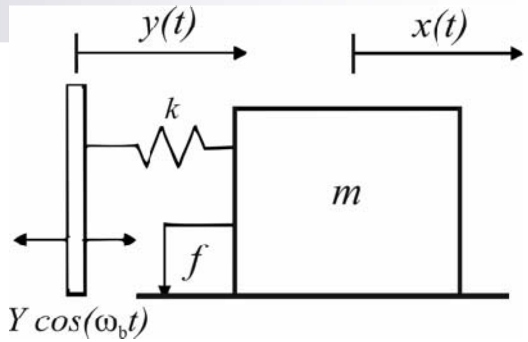
Results



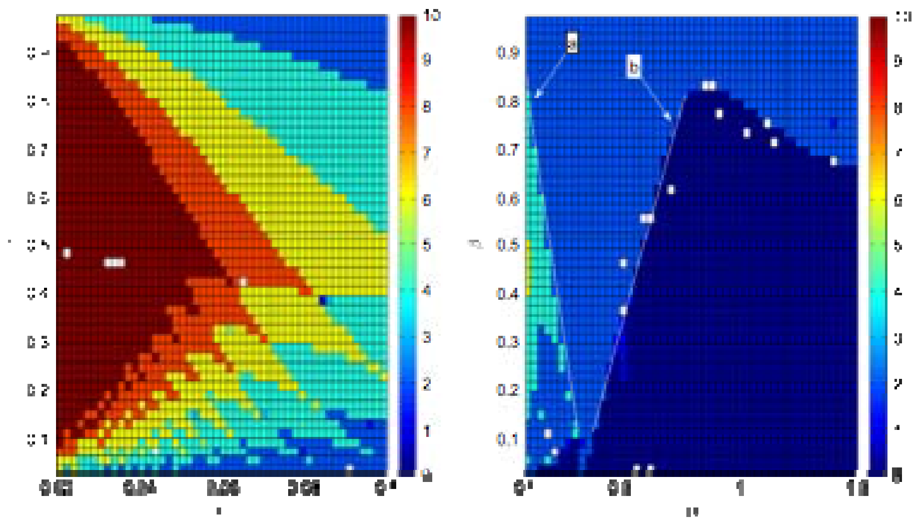
- Considering the parameter space
- Friction and amplitude considered separately
  - Frequency ratio 0.5
- Contours of equal response amplitude



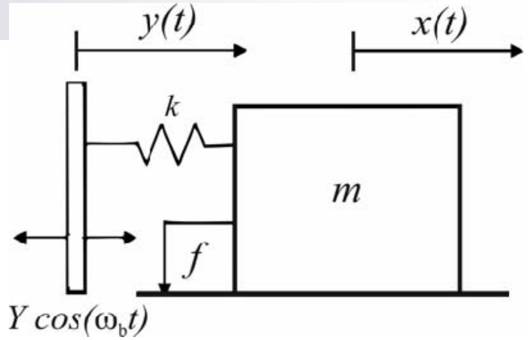
What we are planning to do



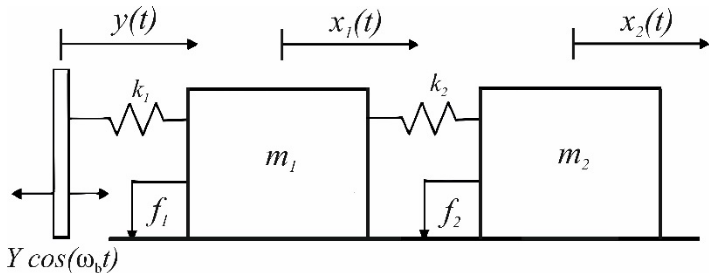
- Which conditions give rise to which boundaries
  - Multiple stop conditions
  - Different conditions describing different parts of the boundary
    - Annotated version of Ciavarella plot



What we are planning to do

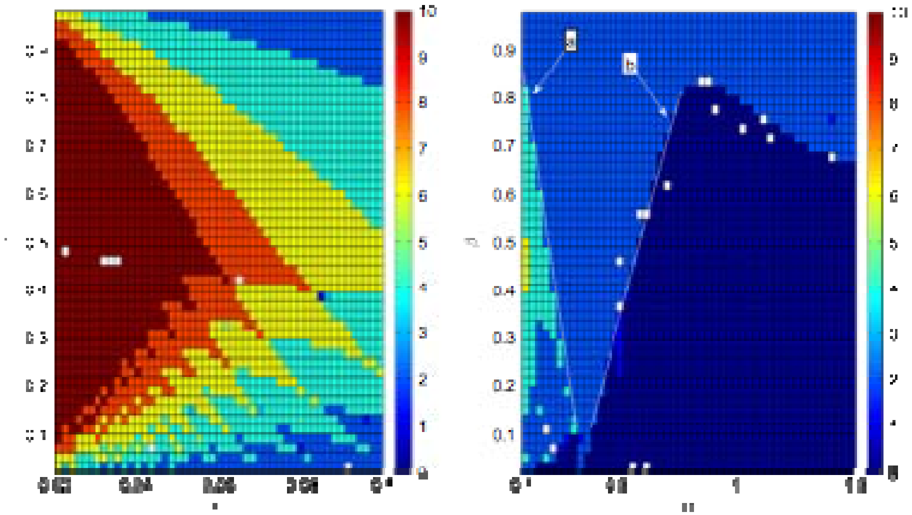


- Closed form solution for no stop case
- Analytic time marching
  - Determining analytic constants
  - Applying non-numeric solving methods
- Multiple degrees of freedom



What we know

- We can get any even number of stops



- We have a symmetric phase portrait
- No evidence of 'non-periodic solutions'
- Oscillations occur 'at the natural frequency during the slip phases'

On the limits of quasi-static analysis for a simple Coulomb frictional oscillator in response to harmonic loads, Papangelo and Ciavarella, 2014

1.1 General properties of contacts

Prof. D.A. Hills

(with Hendrik Andresen)

Department of Engineering Science

University of Oxford

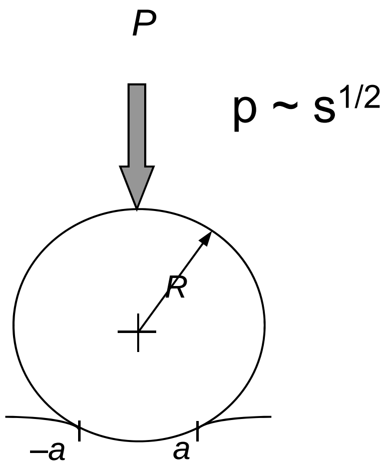
January 2018

"In WP1, research will aim at improving the understanding of the physics of friction contacts in order to develop and validate advanced models for dynamic simulations of turbomachinery models. There is a strong need for advanced contact models, since state-of-the-art contact models are unsatisfactory: when experimental validation is performed, an intense tuning of the models is necessary for numerical predictions to accurately match the experimental results. It is the symptom of a lack of knowledge in the contact behaviour."

Contents

Types of contacts (1a)

1. (a) Incomplete and non-conformal a<<R



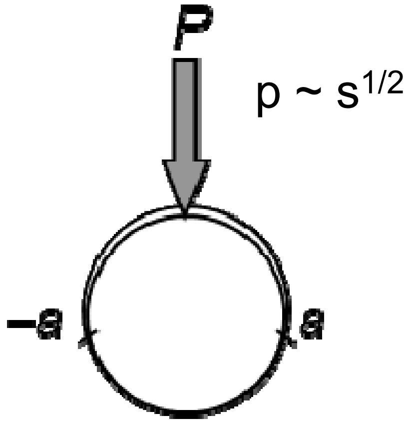
Examples for incomplete contacts:

- roller bearing (picture)
- gear teeth flanks
- convex cam shaft
- rail way wheels

- contact width increases with applied load
- contacting bodies have a common tangent at the contact edges
- contact pressure falls continuously to zero at the contact edges
- both bodies may be represented by half-planes

Types of contacts (1b)

1. (a) Incomplete but conformal a~ R



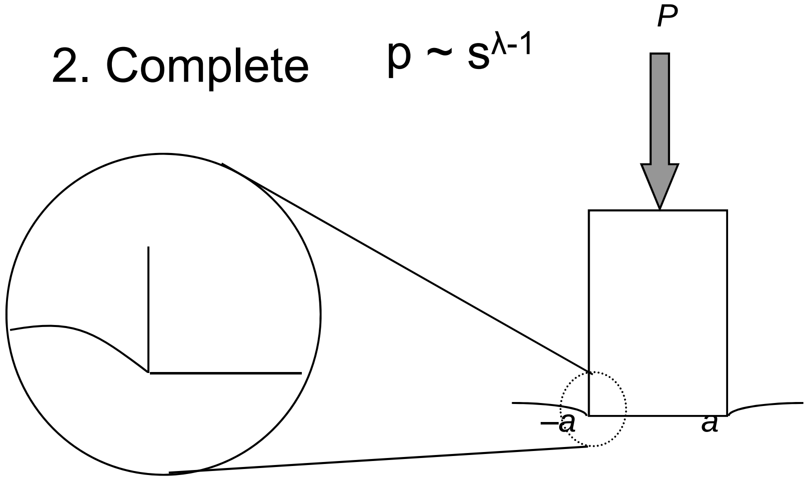
Examples for incomplete but conformal contacts:

- bearing bolt (picture)
- journal bearing

- bodies require formulation for domain shape, e.g. disk and anti-disk

Types of contacts (2)

2. Complete $p \sim s^{\lambda-1}$



Analysis by wedge theory

Examples for complete contacts:

• shipping container on the ground (picture)

• oil drum on the ground

• electric motor on a pedestal

• indenting punch has sharp corners

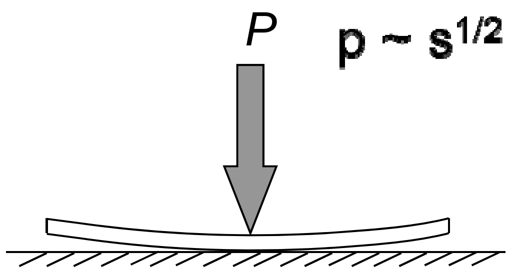
• contact width is fixed by the size of the punch; independent of the contacting load

• relative curvature of the bodies is discontinuous at the contact edges: singular contact pressure results

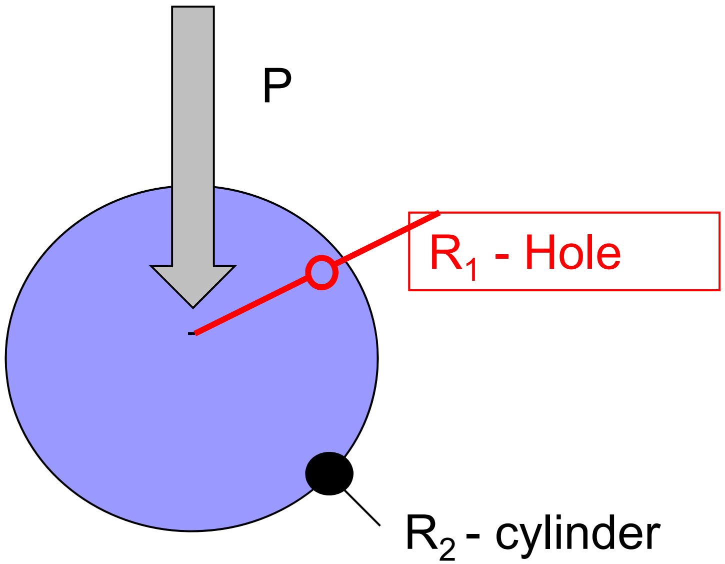
• idealization by a half-plane is certainly inappropriate

Types of contacts (3)

3. Receding



Contact will 'snap' to reduced contact size upon application of very small load



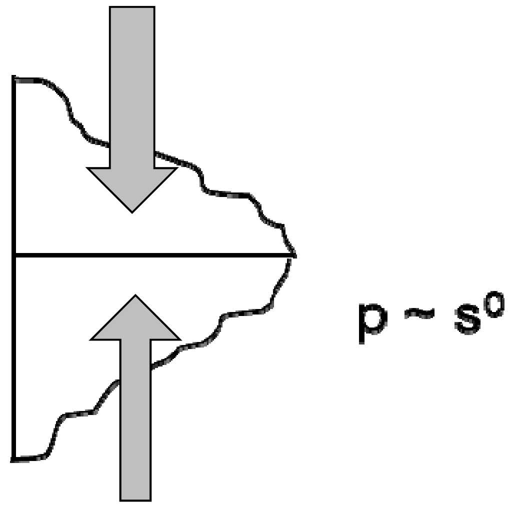
Examples for receding contacts:

- flanged joint (picture)
- bolt array
- high strength friction grip

a. If $R_2 < R_1$ incomplete
b. If $R_2 > R_1$ stationary → smooth recede
c. If $R_2 = R_1$ snap recede

Types of contacts (4)

4. 'Common Edge'



Examples for 'common edge' contacts:

- connecting rod (picture)
- equal flanges

- Both bodies simultaneously define the edge of the contact
- The only kind of contact which has a finite contact pressure at the edge

The point-wise concept of Friction

$$f = \frac{Q}{P} \quad - \textit{concept of sliding friction}$$

$P = \int p(x)dx \quad Q = \int q(x)dx$ - equilibrium

$$f = \frac{Q}{P} = \frac{\int q(x)dx}{\int p(x)dx}$$

Consistent with: $f = \frac{q(x)}{p(x)}$ in regions of slip

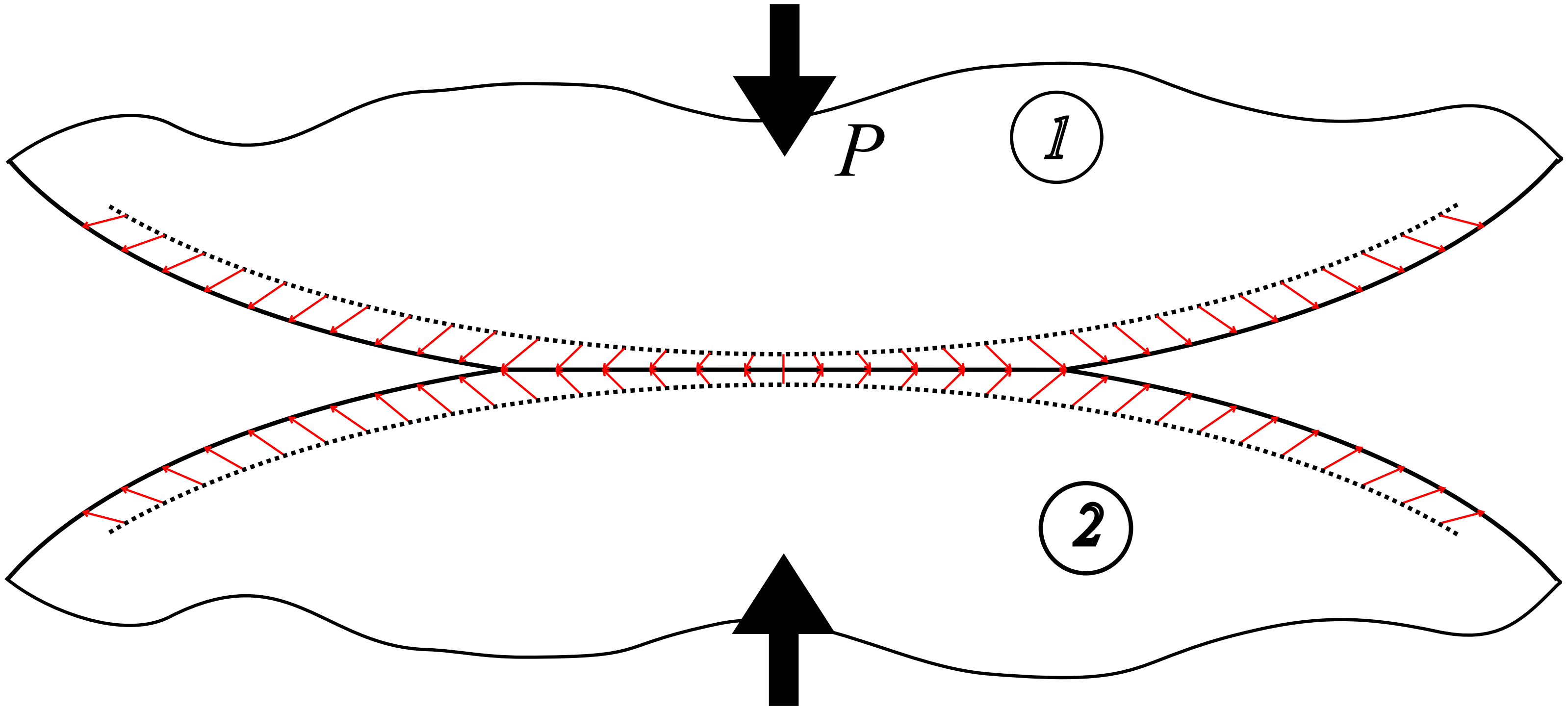
Partial Slip Conditions

- Some regions will stick.
- $|q(x)| < fp(x)$ & $u(x)_1 - u(x)_2$ preserved
- Some regions will slip.
- $|q(x)| = fp(x)$ & sgn(q(x))=sgn($\dot{u}(x)_1 - \dot{u}(x)_2$)

1. We may sometime use strains rather than displacements.

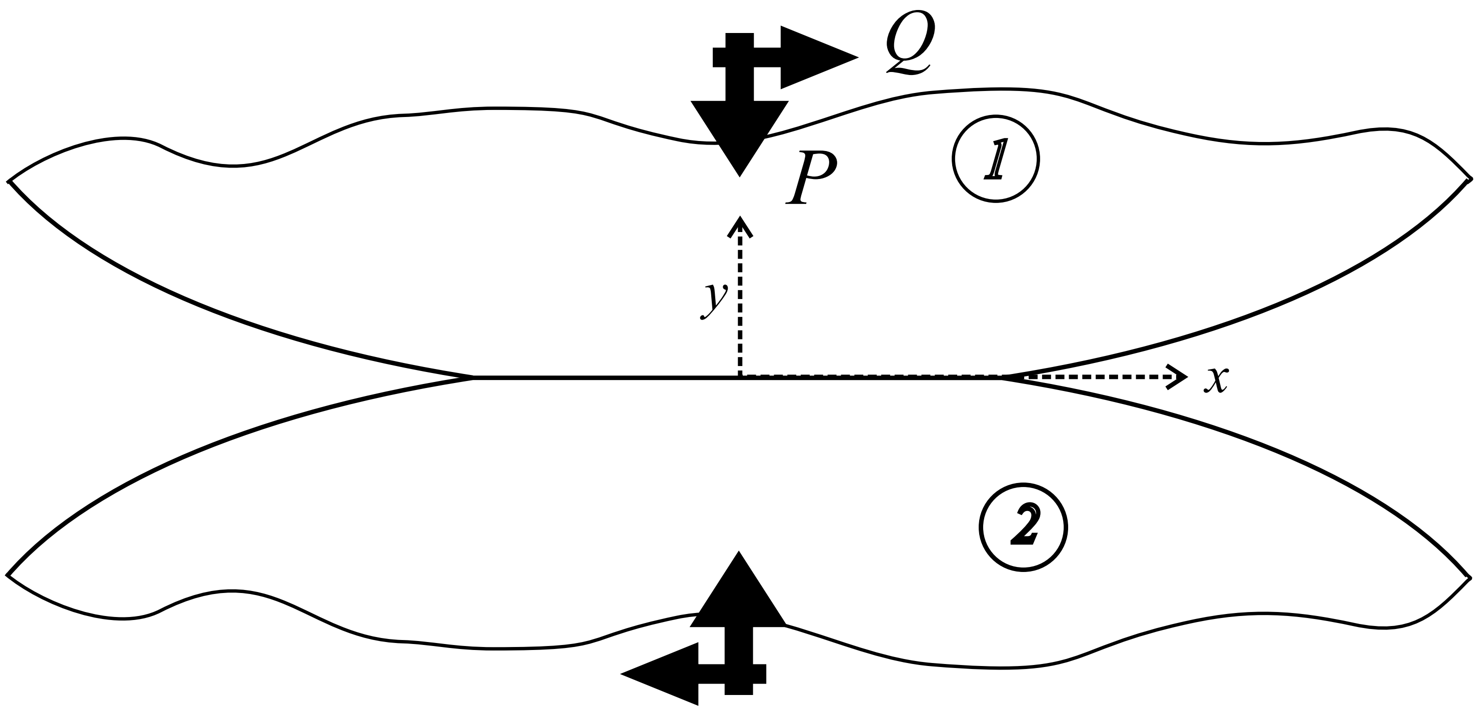
2. Frictional contact are history dependent. A knowledge of the current loads alone is insufficient to be able to specify the slip/stick state.

Normal load – ‘convection’



1. Normal loads will cause surface particles to displace.
2. If the bodies have the same elastic constants points on bodies move by same amount → no surface shear tractions
3. If materials are different → anti-symmetric surface shear [coupling]

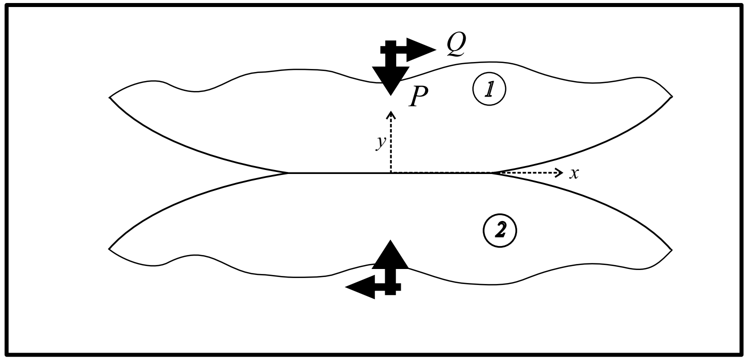
Transverse Loading



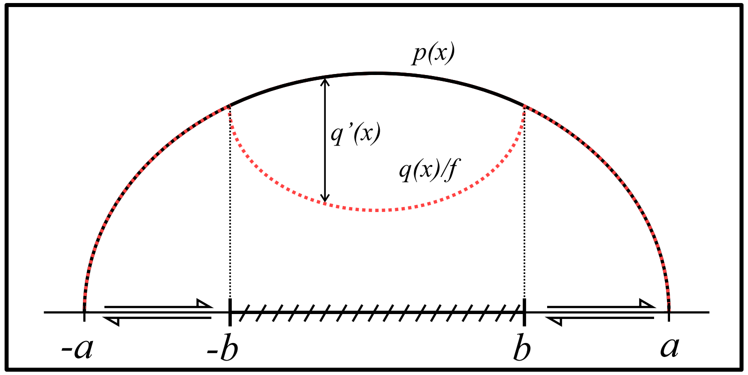
Sequential application of shear forces on formed contact.

Note shear applied in plane of contact (or moment results)

Partial slip solution



Contact geometry



Contact tractions

$$q(x) = fp(x) \quad b \le |x| \le a$$

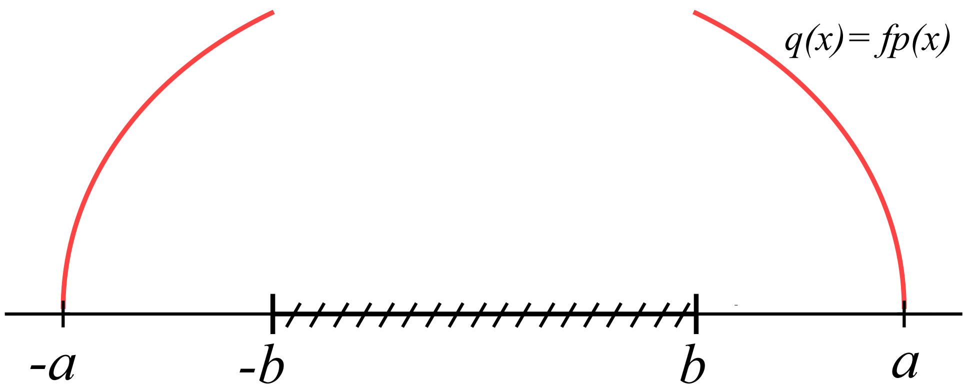
$$\frac{du}{dx} = 0 \quad |x| \le b$$

Conditions for partial slip

Conditions for partial slip

$$q(x) = fp(x) \quad b \leq |x| \leq a$$

$$\frac{du_1}{dx} - \frac{du_2}{dx} = 0 \quad |x| \leq b$$

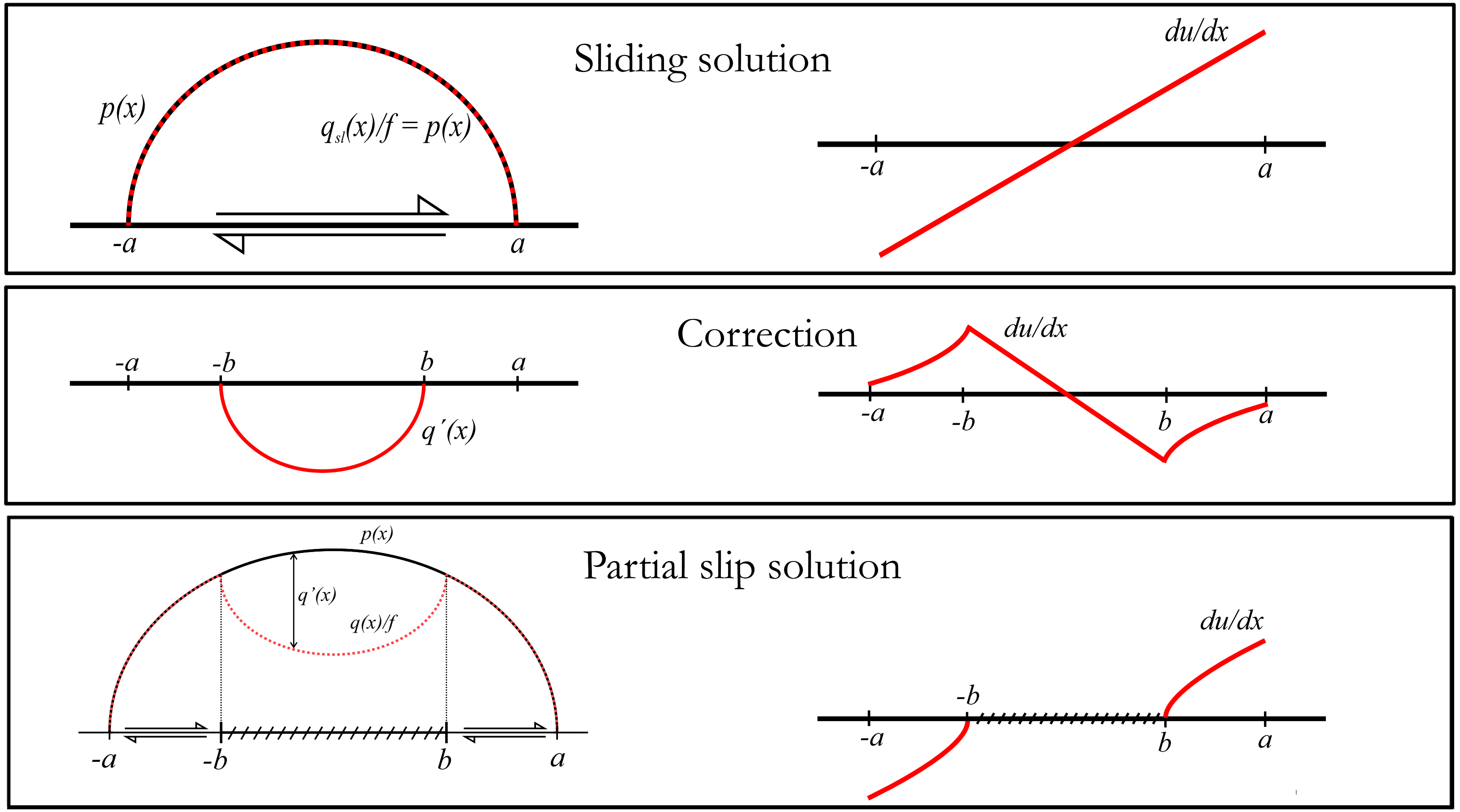


Ciavarella – Jäger theorem (shown for Hertz)

Superposition of two Sliding solutions to achieve a Partial Slip solution

Tractions

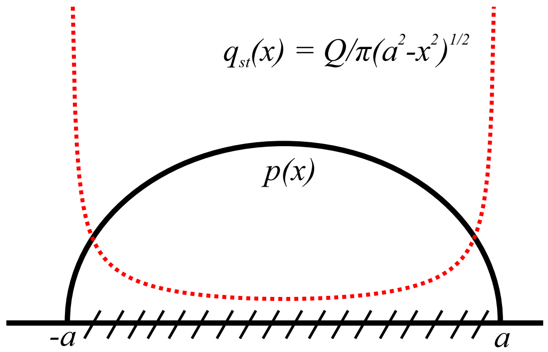
Surface displacements



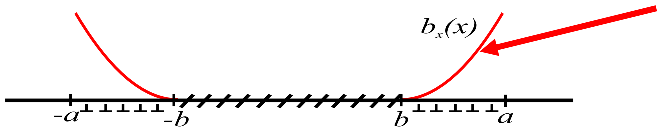
Recent alternative: start from a fully stuck solution

Adhered solution

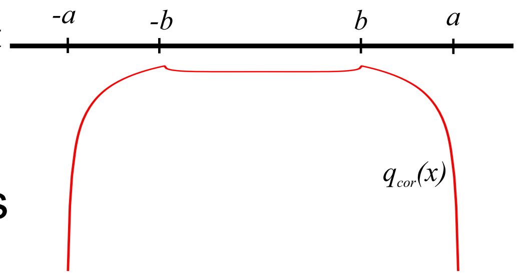
$$\frac{du_1}{dx} - \frac{du_2}{dx} = 0 \quad |x| \leq a$$



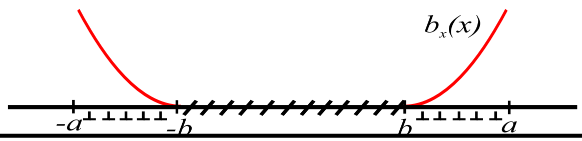
Corrective displacement

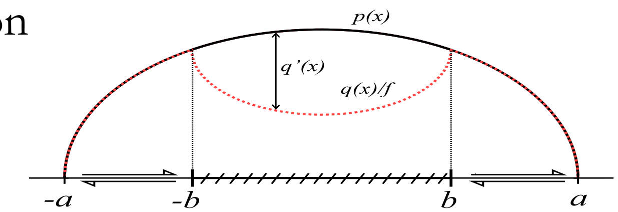


Distribute dislocations

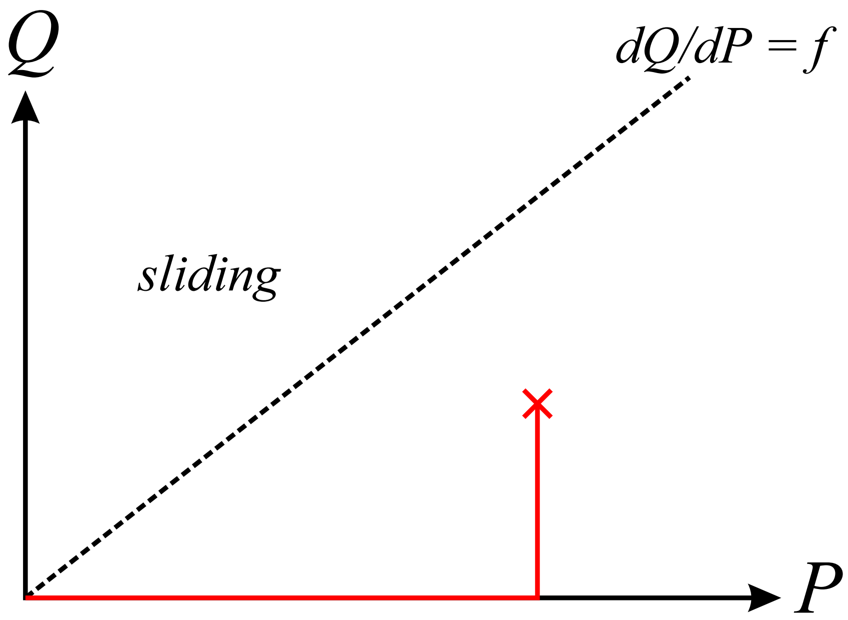


Partial slip solution

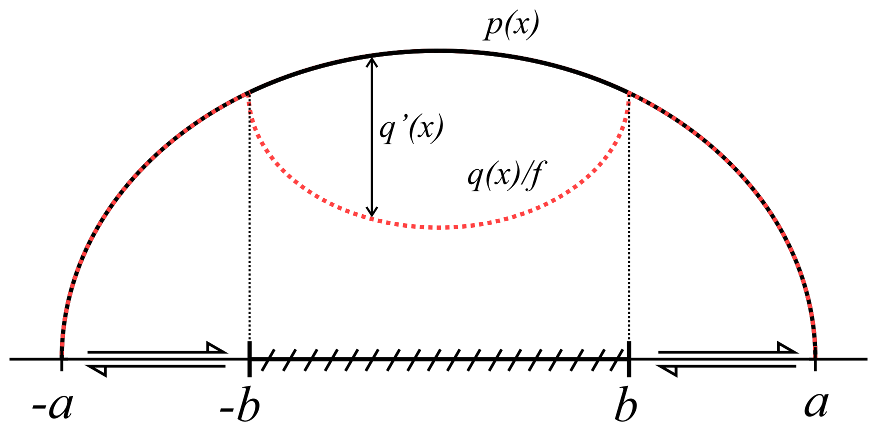




Sequential Loading

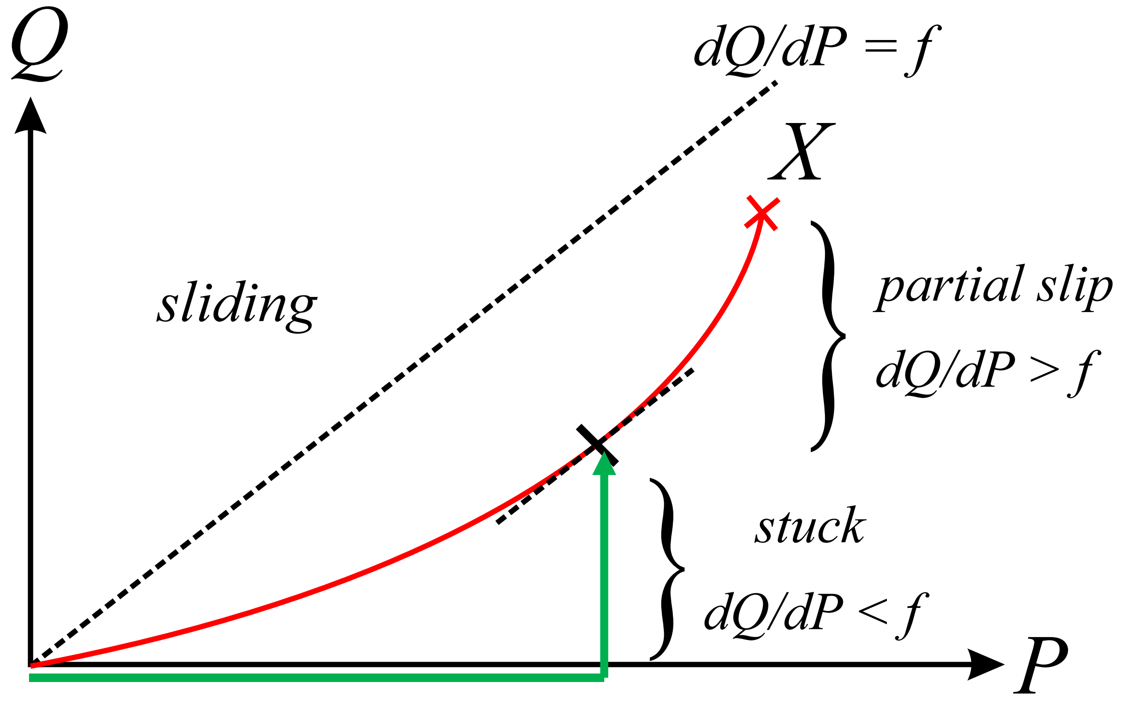


Loading trajectory



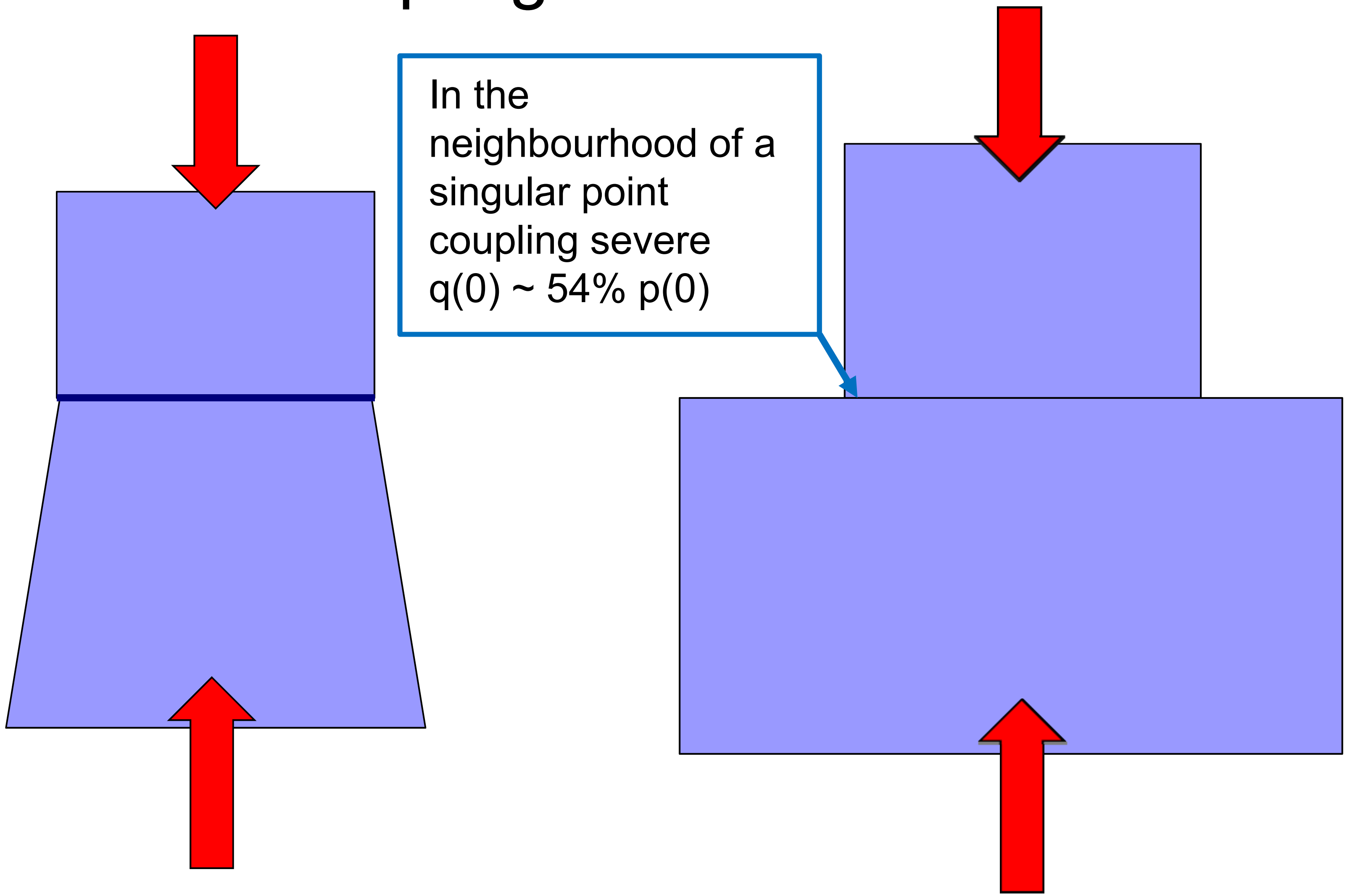
Shear tractions

Varying normal load

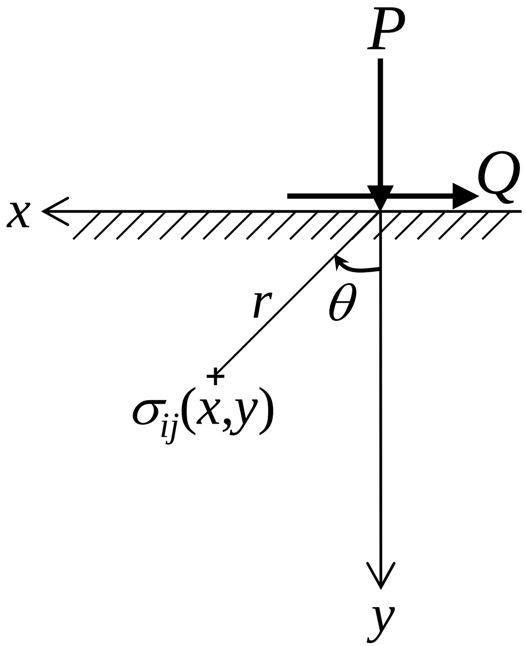


This trajectory would give partial slip

Geometric Coupling



Any difference in elasticity domain shape will cause *some* coupling



First steps in Half-Plane Formulation – the Flamant Solution

- Appropriate Airy stress function (Timoshenko and Goodier, 1950)

$$\phi = -\frac{r\theta}{\pi}\left(P\sin\theta + Q\cos\theta\right)$$

- Complete state of stress in polar coordinates is given by

$$\sigma_{rr} = -\frac{2}{\pi r}\left(P\cos\theta - Q\sin\theta\right)$$

$$\sigma_{\theta\theta} = \sigma_{r\theta} = 0$$

- These stresses can be expressed in Cartesian coordinates as

$$\sigma_{xx} = -\frac{2}{\pi y}\left(P\sin^2\theta\cos^2\theta - Q\sin^3\theta\cos\theta\right)$$

$$\sigma_{yy} = -\frac{2}{\pi y}\left(P\cos^4\theta - Q\sin\theta\cos^3\theta\right)$$

$$\sigma_{xy} = -\frac{2}{\pi y}\left(P\sin\theta\cos^3\theta - Q\sin^2\theta\cos^2\theta\right)$$

where

$$\sin\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$$\cos\theta = \frac{y}{\sqrt{x^2+y^2}}$$

In plane strain:

$$\varepsilon_{xx} = \frac{1}{E}\left\{\sigma_{xx}(1-\nu^2) + \sigma_{yy}\nu(1+\nu)\right\}$$

$$\varepsilon_{yy} = \frac{1}{E}\left\{\sigma_{yy}(1-\nu^2) + \sigma_{xx}\nu(1+\nu)\right\}$$

$$\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

• Surface displacements in Cartesian coordinates (set $\theta = \pm\ \pi/2$)

$$u(x) = -P\left[\frac{(1-2\nu)(1+\nu)}{2E}\right]\mathrm{sgn}(x) + Q\left[\frac{2(1-\nu)(1+\nu)}{\pi E}\right]\ln|x| + \frac{c_1(1+\nu)}{E}$$

$$v(x) = -P\left[\frac{2(1-\nu)(1+\nu)}{\pi E}\right]\ln|x| - Q\left[\frac{(1-2\nu)(1+\nu)}{2E}\right]\mathrm{sgn}(x) + \frac{c_2(1+\nu)}{E}$$

where

$$\mathrm{sgn}(x) = 1 \quad \text{if} \quad x > 0$$

$$\mathrm{sgn}(x) = -1 \quad \text{if} \quad x < 0$$

Integrate strains to get displacement field and then specialise to surface

• Note that the displacements are not absolute quantities

• Therefore it is difficult to use the equations above to formulate the contact problem

• A better method is to relate the surface loads to the *gradient* of the surface

Normal displacement gradient

$$\frac{\partial v}{\partial x} = \frac{2(1-\nu)(1+\nu)}{\pi E} \int_{\text{contact}} \frac{p(\xi)d\xi}{x-\xi}$$ due to normal pressure

$$\frac{\partial v}{\partial x} = -\frac{(1-2\nu)(1+\nu)}{E} q(x)$$ due to shearing traction

Note *q(x)* affects only slope at *x*; and *p(x)* affects only strain at *x*.

Shear displacement gradient

$$\frac{\partial u}{\partial x} = \frac{(1-2\nu)(1+\nu)}{E} p(x)$$ due to normal pressure

$$\frac{\partial u}{\partial x} = \frac{2(1-\nu)(1+\nu)}{\pi E} \int \frac{q(\xi)d\xi}{x-\xi}$$ due to shearing traction

Total normal and shear displacement gradients

- Adding together the two effects gives the following pair of equations:

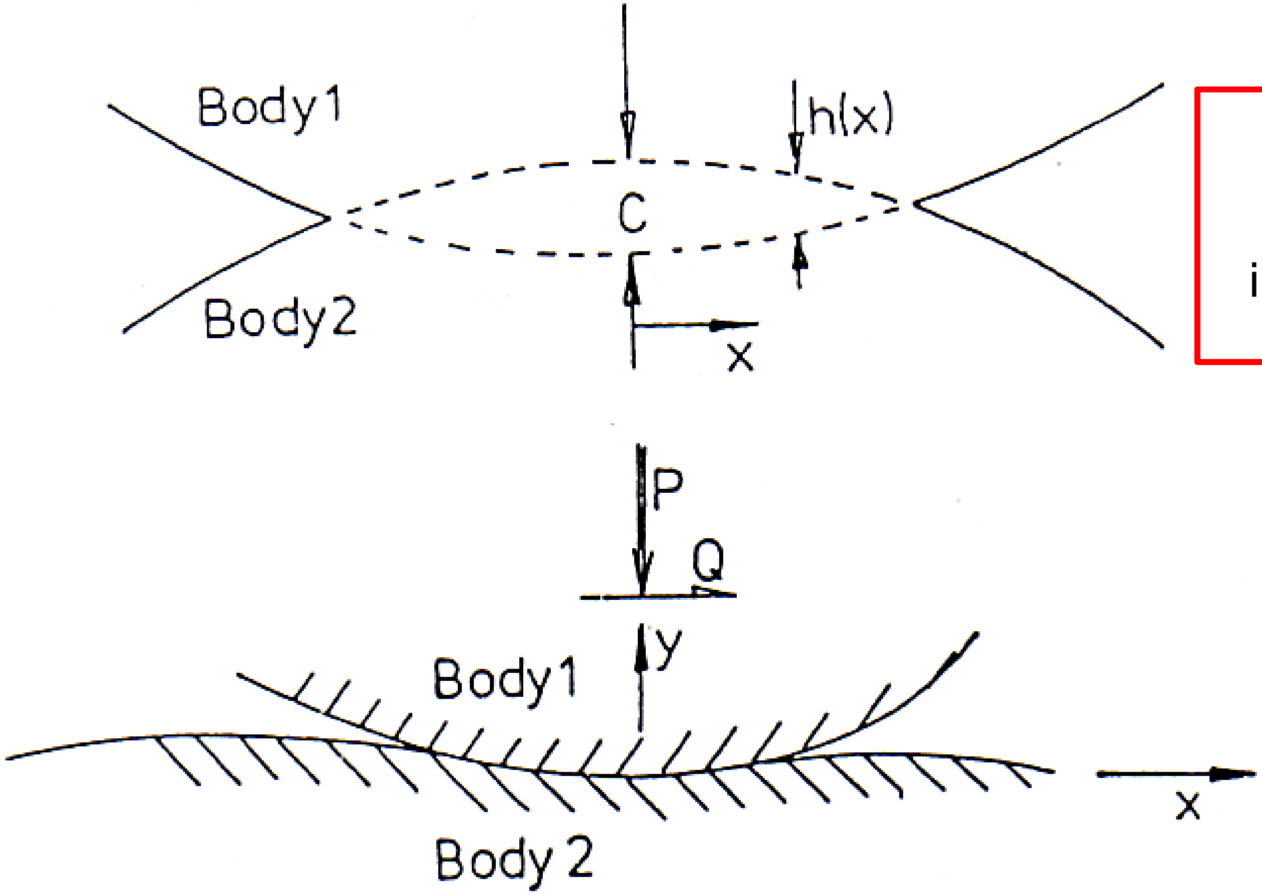
$$\frac{\partial v}{\partial x} = \frac{2(1-\nu)(1+\nu)}{\pi E}\int \frac{p(\xi)d\xi}{x-\xi} - \frac{(1-2\nu)(1+\nu)}{E} q(x)$$

$$\frac{\partial u}{\partial x} = \frac{(1-2\nu)(1+\nu)}{E} p(x) + \frac{2(1-\nu)(1+\nu)}{\pi E}\int \frac{q(\xi)d\xi}{x-\xi}$$

- This pair of equations provides a useful set of influence functions for the loading of a half plane.

Note that surface pressure produces not only surface depression but also in-plane strain

General plane contact problem formulation



If the bodies could interpenetrate

• Both of the contacting bodies are deformable. Therefore we need to establish the effect of the interface tractions on the combined deformation.

• For generality, assume that the contacting bodies have different elastic constants.

• The normal surface displacements are:

$$\frac{\partial v_1}{\partial x} = \frac{2(1-\nu_1)(1+\nu_1)}{\pi E_1} \int \frac{p(\xi)d\xi}{x-\xi} - \frac{(1-\nu_1)(1+\nu_1)}{E_1} q(x) \quad \text{(Body 1)}$$

$$\frac{\partial v_2}{\partial x} = -\frac{2(1-\nu_2)(1+\nu_2)}{\pi E_2} \int \frac{p(\xi)d\xi}{x-\xi} - \frac{(1-\nu_2)(1+\nu_2)}{E_2} q(x) \quad \text{(Body 2)}$$

Note sign change

• Suppose that the two bodies could freely interpenetrate each other, and the amount of overlap is $h(x)$.

• The two bodies must now deform such that the relative surface normal displacement, $v_1(x) - v_2(x) = h(x)$.

• Therefore

$$\frac{1}{A}\frac{\partial h}{\partial x} = \frac{1}{\pi} \int \frac{p(\xi)d\xi}{x-\xi} - \beta q(x)$$

where the compliance of the bodies and the influence of their mismatch for plane strain, are respectively

$$A = 2\left\{\frac{(1-\nu_1^2)}{E_1} + \frac{(1-\nu_2^2)}{E_2}\right\}$$

Composite Compliance (plane strain)

$$\beta = \frac{\dfrac{(1+\nu_1)(1-2\nu_2)}{E_1} - \dfrac{(1+\nu_2)(1-2\nu_2)}{E_2}}{\dfrac{(1-\nu_1^2)}{E_1} + \dfrac{(1-\nu_2^2)}{E_2}}$$

Dundurs' second constant

• Similarly, the relative surface tangential displacement, $u_1(x) - u_2(x) = g(x)$. Giving,

$$\frac{1}{A}\frac{\partial g}{\partial x} = \frac{1}{\pi}\int \frac{q(\xi)d\xi}{x-\xi} + \beta\, p(x)$$

Pressure prodcues surface strains if β≠0

• Relative surface normal & tangential displacement equations will enable any plane contact problem to be solved.

• Now, consider relative surface normal displacement equation. The effect of shearing traction will vanish if either:

1. The contact is perfectly lubricated, or

2. Dundurs' constant, $\beta = 0$ ⟶ normally due to elastically similar bodies.

• If either of the conditions above is fulfilled, the following simplified equation results

$$\frac{1}{A}\frac{\partial h}{\partial x} = \frac{1}{\pi}\int_{-a}^{a}\frac{p(\xi)d\xi}{x-\xi}$$

• This is a Cauchy singular integral equation of the first kind.

• The unknown quantity, $p(x)$, is contained within the integral, and an inversion formula is needed. See Appendix B.

Example 1 : Hertzian contact

- The surface profile can be idealized as a parabola, giving

$$h(x) = C - \frac{1}{2}kx^2$$

where

$$k = \frac{1}{R_1} + \frac{1}{R_2}$$

R_1, R_2 are the radii of curvature of the cylinders

- Relative surface normal displacement is now written as

$$\frac{1}{\pi}\int_{-a}^{a} \frac{p(\xi)d\xi}{x-\xi} = -\frac{kx}{A}$$

- Normalise the interval of integration by making the substitutions

$$\xi = ar \quad \text{and} \quad x = as$$

so that

$$\frac{1}{\pi}\int_{-1}^{1}\frac{p(r)dr}{r-s} = \frac{ka}{A}s \qquad -1 \le s \le 1$$

- Using the inversion formula

$$p(s) = -\frac{1}{\pi}\sqrt{1-s^2}\left(\frac{ka}{A}\right)\int_{-1}^{1}\frac{r\,dr}{\sqrt{1-r^2}(r-s)}$$

$$= -\left(\frac{ka}{A}\right)\sqrt{1-s^2}$$

- Note that the consistency equation is automatically satisfied.

- Contact half-width, a may be found by

$$P = -a\int_{-a}^{a} p(s)ds = \frac{k\pi a^2}{2A}$$

so that

$$a^2 = \frac{2AP}{\pi k}$$

and

$$p(x) = -\frac{2P}{\pi a}\sqrt{1-(x/a)^2}$$
$$= p_o\sqrt{1-(x/a)^2}$$

Example 2 : Indentation by a Punch

- Consider the indentation of a half-plane by a rigid flat-ended punch

- The value of the compliance is

$$A = \frac{1-\nu_2}{\mu_2} = \frac{2(1-\nu_2^2)}{E_2}$$

- The punch-end is flat ⟶ slope is zero

$$\frac{dh}{dx} = 0$$

- Relative surface normal displacement is now written as

$$\frac{A}{\pi}\int_{-1}^{1}\frac{p(r)dr}{s-r} = 0$$

- Singular solution at both ends is expected

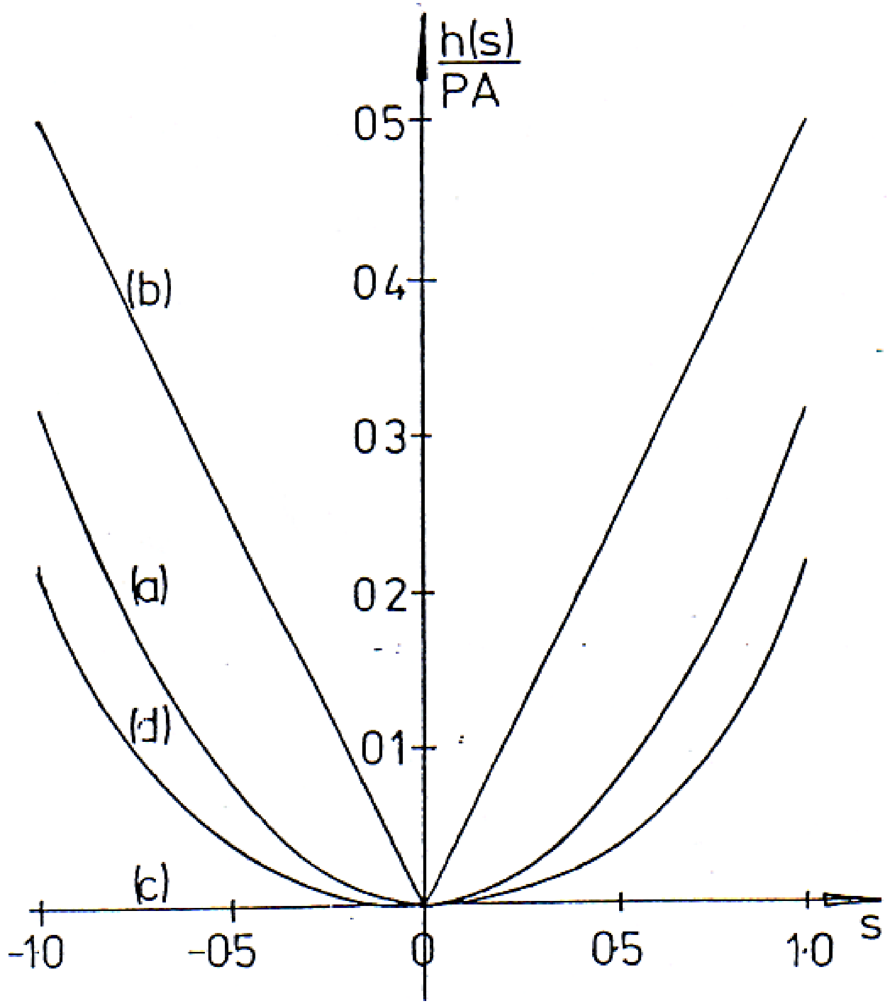
- Using the inversion formula

$$p(s) = \frac{C}{\sqrt{1-s^2}}$$

- The value of the constant is found by ensuring vertical equilibrium, giving

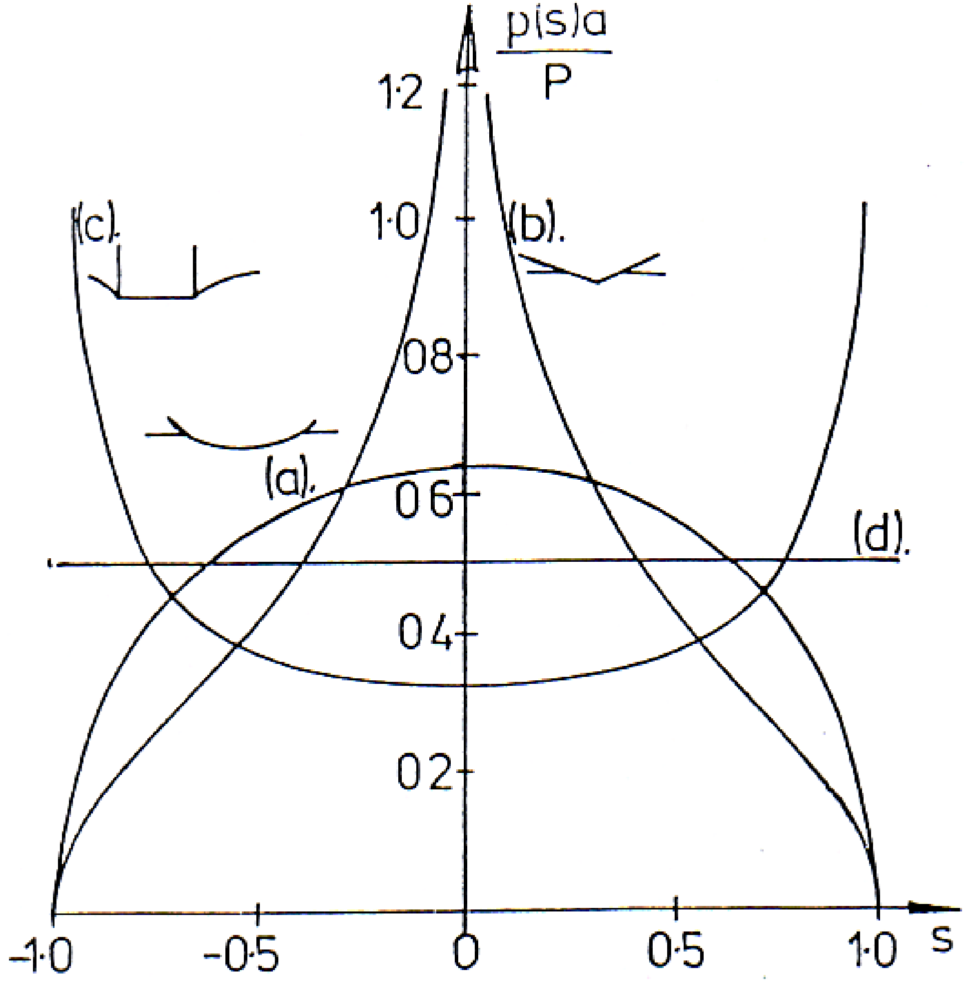
$$p(s) = -\frac{P}{a\pi\sqrt{1-s^2}}$$

A comparison of indenter profiles



(a) Hertzian contact (b) Wedge indentation
(c) Flat ended punch (d) Punch generating uniform pressure

Resulting pressure distributions



(a) Hertzian contact (b) Wedge indentation
(c) Flat ended punch (d) Punch generating uniform pressure

The ‘learning outcome’ of this lecture is that;

(a) You should appreciate the principal different kinds of contact behaviour

(b) You should understand the concept of ‘coupling’, and where it is severe, where it is mild

(c) You should begin to understand what a partial slip contact is

(d) You should have begin to understand how to solve the normal aspect of a ‘half-plane’ contact problem (especially the classical Hertz problem)

Appendix A: Airy Stress Formulation (polar coordinates)

- The Airy stress function – stress relationships are:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2}$$

$$\sigma_{r\theta} = -\frac{1}{r}\frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \phi}{\partial \theta}$$

whilst the biharmonic (compatibility) equation is

$$\nabla^4 \phi = \nabla^2 \nabla^2 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \right)^2 \phi = 0.$$

Appendix B: Summary of the inversion formula (First Kind)

- Equation to be solved:

$$\frac{1}{\pi}\int_{-a}^{a}\frac{f(s)}{s-x}ds = g(x), \qquad |x| < a$$

- Solution:

$$f(x) = -\frac{w(x)}{\pi}\int_{-a}^{a}\frac{g(s)\,ds}{w(s)(s-x)} + C\,w(x)$$

- The solution is required to have the following characteristics:

1. Singular at both end points, $x = \pm a$

$$w(x) = \frac{1}{\sqrt{a^2 - x^2}} \qquad C \neq 0$$

2. Non singular at $x = a$

$$w(x) = \sqrt{\frac{a-x}{a+x}} \qquad C = 0$$

3. Non singular at $x = -a$

$$w(x) = \sqrt{\frac{a+x}{a-x}} \qquad C = 0$$

4. Non singular at both end points, $x = \pm a$

$$w(x) = \sqrt{a^2 - x^2} \qquad C = 0$$

with the consistency condition

$$\int_{-a}^{a} \frac{g(s)}{w(s)} ds = 0$$